Math 2040 C Linear Algebra II

Recall: Vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$

- Vector addition, scalar multiplication

  $\rightsquigarrow$ Subspace, linear independence, Span, basis, dimension, ...

  eg $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ is a basis of 2-dimension subspace in $\mathbb{R}^3$

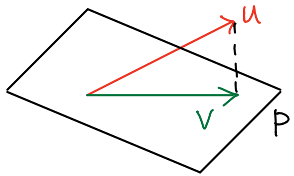
- Matrix

  $\rightsquigarrow$ Kernal, range, eigenvalues, diagonalization

- Inner product $\langle u, v \rangle$

  $\rightsquigarrow$ Length, Angle

  Orthogonal projection



v is the point on P closest to u

We will generalize these ideas from $\mathbb{R}^n$ to other sets. eg. polynomials, functions

eg. The solution space of the differential equation

$$f''(t) = f'(t) + 6f(t)$$

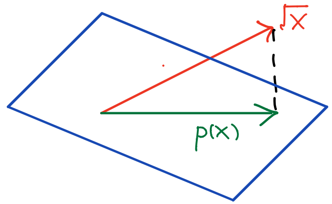
is "2-dimensional" with basis $\{e^{3t}, e^{-2t}\}$

eg linear polynomial

$\{ax+b : a, b \in \mathbb{R}\} \subseteq \{\text{all functions}\} \ni \sqrt{x}$

dim = 2 dim = ∞

Using an appropriated inner product and projection



subspace of linear polynomials

Computation $\Rightarrow$ $p(x) = \frac{4}{5}x + \frac{4}{15}$ is the "best" linear polynomial to approximate $\sqrt{x}$ on $[0, 1]$

Vector space

Let $\mathbb{F}$ be a field (something with $+, -, \times, \div$)

e.g. $\mathbb{F} = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \cancel{\mathbb{Z}}$

($2, 3 \in \mathbb{Z}$ but $\frac{2}{3} \notin \mathbb{Z}$: cannot do division within $\mathbb{Z}$)

For simplicity, we may assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Defn A vector space over $\mathbb{F}$ is a set V together with two operations

i. Addition $\quad + : V \times V \longrightarrow V$

$$(u, v) \longmapsto u + v$$

ii Scalar Multiplication $\quad \cdot : \mathbb{F} \times V \longrightarrow V$

$$(\lambda, v) \longmapsto \lambda v$$

such that properties (VS1) – (VS7) hold

Rmk
$\forall$ means for all
$\exists$ means there exist

(VS1) Commutativity

$$\forall u, v \in V, \quad u + v = v + u$$

(VS2) Additive Associativity

$$\forall u, v, w \in V \quad (u + v) + w = u + (v + w)$$

(VS3) Additive Identity

$\exists$ an element $\vec{0} \in V$ such that $v + \vec{0} = v \quad \forall v \in V$

called zero vector

(VS4) Additive Inverse

$\forall v \in V, \exists w \in V$ such that $v + w = \vec{0}$

called additive inverse of v

(VS5) Multiplicative Identity

$\forall v \in V, \quad 1 v = v \quad$ (Here $1 \in \mathbb{F}$)

(VS6) Multiplicative Associativity

$$\forall a, b \in \mathbb{F} \quad v \in V, \quad (ab) v = a (bv)$$

(VS7) Distributive Properties

i $\quad \forall a \in \mathbb{F}, u, v \in V \quad a(u + v) = au + av$

ii $\quad \forall a, b \in \mathbb{F} \quad v \in V \quad (a + b) v = av + bv$

Rmk

- If $\mathbb{F} = \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$, then V is called a $\begin{cases} \text{real} \\ \text{complex} \end{cases}$ vector space

- Elements of V are called **vectors**

  Elements of $\mathbb{F}$ are called **scalars**

eg

Let $P_m(\mathbb{F}) = \{a_0 + a_1 x + \cdots + a_m x^m : a_i \in \mathbb{F} \ \forall i\}$

be the set of polynomials with coefficients in $\mathbb{F}$

and degree $\leq m$.

Verify that $P_m(\mathbb{F})$ with the usual addition and

scalar multiplication

$$\left(\sum_{k=0}^{m} a_k x^k\right) + \left(\sum_{k=0}^{m} b_k x^k\right) = \sum_{k=0}^{m} (a_k + b_k) x^k$$

$$\lambda\left(\sum_{k=0}^{m} a_k x^k\right) = \sum_{k=0}^{m} (\lambda a_k) x^k$$

form a vector space over $\mathbb{F}$

Sol Need to verify (VS1)-(VS7) holds

Suppose $a, b \in \mathbb{F}$ and $u, v, w \in P_m(\mathbb{F})$ with

$$u = \sum_{k=0}^{m} a_k x^k, \quad v = \sum_{k=0}^{m} b_k x^k, \quad w = \sum_{k=0}^{m} c_k x^k.$$

(VS1)
$$\begin{aligned} u + v &= \left(\sum_{k=0}^{m} a_k x^k\right) + \left(\sum_{k=0}^{m} b_k x^k\right) \\ &= \sum_{k=0}^{m} (a_k + b_k) x^k \\ &\overset{(*)}{=} \sum_{k=0}^{m} (b_k + a_k) x^k \\ &= \left(\sum_{k=0}^{m} b_k x^k\right) + \left(\sum_{k=0}^{m} a_k x^k\right) \end{aligned}$$

Rmk As seen above, additive commutativity of $P_m(\mathbb{F})$ follows from that of $\mathbb{F}$ $(*)$.

(VS 2)

$$(u+v)+w = \left[\left(\sum_{k=0}^{m} a_k x^k\right) + \left(\sum_{k=0}^{m} b_k x^k\right)\right] + \left(\sum_{k=0}^{m} c_k x^k\right)$$

$$= \left(\sum_{k=0}^{m} (a_k + b_k) x^k\right) + \left(\sum_{k=0}^{m} c_k x^k\right)$$

$$= \sum_{k=0}^{m} \left[(a_k + b_k) + c_k\right] x^k$$

$$= \sum_{k=0}^{m} \left[a_k + (b_k + c_k)\right] x^k$$

$$= \left(\sum_{k=0}^{m} a_k x^k\right) + \left[\sum_{k=0}^{m} (b_k + c_k) x^k\right]$$

$$= \left(\sum_{k=0}^{m} a_k x^k\right) + \left[\left(\sum_{k=0}^{m} b_k x^k\right) + \left(\sum_{k=0}^{m} c_k x^k\right)\right]$$

$$= u + (v + w)$$

(VS 3) Let $\vec{0} = \left(\sum_{k=0}^{m} 0 x^k\right)$. Then

$$u + \vec{0} = \left(\sum_{k=0}^{m} a_k x^k\right) + \left(\sum_{k=0}^{m} 0 x^k\right)$$

$$= \left(\sum_{k=0}^{m} (a_k + 0) x^k\right)$$

$$= \left(\sum_{k=0}^{m} a_k x^k\right)$$

$$= u$$

(VS 4) Consider $\sum_{k=0}^{m} (-a_k) x^k \in P_m(\mathbb{F})$

$$u + \left(\sum_{k=0}^{m} (-a_k) x^k\right) = \left(\sum_{k=0}^{m} a_k x^k\right) + \left(\sum_{k=0}^{m} (-a_k) x^k\right)$$

$$= \sum_{k=0}^{m} \left[a_k + (-a_k)\right] x^k$$

$$= \sum_{k=0}^{m} 0 x^k$$

$$= \vec{0}$$

(VS5) $$1u = 1\left(\sum_{k=0}^{m} a_k x^k\right)$$
$$= \sum_{k=0}^{m} (1a_k) x^k$$
$$= \left(\sum_{k=0}^{m} a_k x^k\right)$$
$$= u$$

(VS6) $$(ab)u = (ab)\left(\sum_{k=0}^{m} a_k x^k\right)$$
$$= \sum_{k=0}^{m} \left[(ab)a_k\right] x^k$$
$$= \sum_{k=0}^{m} \left[a(ba_k)\right] x^k$$
$$= a\left[\sum_{k=0}^{m} (ba_k) x^k\right]$$
$$= a\left[b\left(\sum_{k=0}^{m} a_k x^k\right)\right]$$
$$= a(bu)$$

(VS7) i. $$a(u+v) = a\left[\left(\sum_{k=0}^{m} a_k x^k\right) + \left(\sum_{k=0}^{m} b_k x^k\right)\right]$$
$$= a\left[\sum_{k=0}^{m} (a_k + b_k) x^k\right]$$
$$= \sum_{k=0}^{m} \left[a(a_k + b_k)\right] x^k$$
$$= \sum_{k=0}^{m} (aa_k + ab_k) x^k$$
$$= \sum_{k=0}^{m} (aa_k) x^k + \sum_{k=0}^{m} (ab_k) x^k$$
$$= a\left(\sum_{k=0}^{m} a_k x^k\right) + a\left(\sum_{k=0}^{m} b_k x^k\right)$$
$$= au + av$$

ii. $$(a+b)u = (a+b)\left(\sum_{k=0}^{m} a_k x^k\right)$$
$$= \sum_{k=0}^{m} \left[(a+b)a_k\right] x^k$$
$$= \sum_{k=0}^{m} (aa_k + ba_k) x^k$$
$$= \sum_{k=0}^{m} (aa_k) x^k + \sum_{k=0}^{m} (ba_k) x^k$$
$$= a\left(\sum_{k=0}^{m} a_k x^k\right) + b\left(\sum_{k=0}^{m} a_k x^k\right)$$
$$= au + bu$$

Some other examples of vector spaces

① $\mathbb{F}^n = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{F}\ \forall i\}$ with

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n)$$

$$\lambda(x_1, \cdots, x_n) = (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$$

② $\mathbb{F}^\infty$ = the set of all sequences in $\mathbb{F}$

$$(x_1, x_2, \cdots) + (y_1, y_2, \cdots) = (x_1 + y_1, x_2 + y_2, \cdots)$$

$$\lambda(x_1, x_2, \cdots) = (\lambda x_1, \lambda x_2, \cdots)$$

e.g. $(1,1,1,1,1,1,\cdots) + (1,-1,1,-1,1,-1,\cdots)$

$= (2,0,2,0,2,0,\cdots)$

③ $P(\mathbb{F})$ = the set of all polynomials with coefficients in $\mathbb{F}$

with usual addition and scalar multiplication

④ $M_{m\times n}(\mathbb{F})$ = the set of all $m\times n$ matrices with entries in $\mathbb{F}$

with usual matrix addition and scalar multiplication

⑤ Let S be a set and

$\mathbb{F}^S = \{f : S \to \mathbb{F}\}$

= the set of all functions from S to $\mathbb{F}$

For $f, g \in \mathbb{F}^S$, $\lambda \in \mathbb{F}$, define $f+g$, $\lambda f \in \mathbb{F}^S$ by

$$(f+g)(x) = f(x) + g(x)$$

$$(\lambda f)(x) = \lambda f(x)$$

Rmk Let $S = \{1, 2, 3, \cdots, n\}$. Then $\mathbb{F}^S = \mathbb{F}^n$

Similarly, $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{F}^{\mathbb{N}} = \mathbb{F}^\infty$

Ex Verify that each of ① to ⑤ are vector space over $\mathbb{F}$

Rmk Every complex vector space can also be regarded as a real vector space.

Reason: If V is a complex vector space, then (VS1)-(VS7) in definition of vector space hold for any $a, b \in \mathbb{C}$

$\Rightarrow$ (VS1)-(VS7) also hold for any $a, b \in \mathbb{R}$

$\therefore$ V is also a real vector space

eg. $\mathbb{C}^3 = \{(z_1, z_2, z_3) : z_1, z_2, z_3 \in \mathbb{C}\}$ is a complex vector space (of dim 3) and also a real vector space (of real dimension 6)

Rmk We will define dimension later

eg Determine if $V = \mathbb{R}^2$ with the following (non-standard) addition and scalar multiplication is a real vector space.

i $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$

$\lambda(a_1, a_2) = (\lambda^2 a_1, \lambda^3 a_2)$

ii $(a_1, a_2) + (b_1, b_2) = (a_1 a_2, b_1 b_2)$

$\lambda(a_1, a_2) = (\lambda a_1, \lambda a_2)$

iii $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$

$\lambda(a_1, a_2) = (\lambda a_1, \lambda a_2)$

Sol

(i) Note that

$$2(1,0) + 3(1,0) = (4,0) + (9,0) = (13,0)$$

Also, $(2+3)(1,0) = 5(1,0) = (25,0)$

$\therefore$ $2(1,0) + 3(1,0) \neq (2+3)(1,0)$ $\quad$ $\cancel{\text{(VS7)}}$

$\therefore$ V is not a vector space

(ii), (iii) Exercise

Proposition

Let V be a vector space over $\mathbb{F}$. Then

① The element $\vec{0} \in V$ in (VS3) is unique.
It is called the **zero vector** of V

② For $v \in V$, the element $w \in V$ in (VS4) is unique. It is called the **additive inverse** of v and is denoted by $-v$

③ **(Cancellation law)** Let $u, v, w \in V$
If $u + w = v + w$ then $u = v$

④ $0v = \vec{0}$ for any $v \in V$

$$\underset{\text{scalar}}{0}\, v = \underset{\text{vector}}{\vec{0}}$$

⑤ $a\vec{0} = \vec{0}$ for any $a \in \mathbb{F}$

⑥ $(-1)v = -v$ for any $v \in V$

Pf

① Suppose $\vec{0}, \vec{0}' \in V$ and for any $\vec{v} \in V$

$$\vec{v} + \vec{0} = \vec{v}, \quad \vec{v} + \vec{0}' = \vec{v}$$

Then $\vec{0} = \vec{0} + \vec{0}' = \vec{0}' + \vec{0} = \vec{0}'$

② Let $v \in V$. Suppose $w, w' \in V$ and $v + w = v + w' = \vec{0}$. Then

$$\begin{aligned} w &= w + \vec{0} && \text{(VS 3)} \\ &= w + (v + w') && \text{(assumption)} \\ &= (w + v) + w' && \text{(VS 2)} \\ &= (v + w) + w' && \text{(VS 1)} \\ &= 0 + w' && \text{(assumption)} \\ &= w' + 0 && \text{(VS 1)} \\ &= w' && \text{(VS 3)} \end{aligned}$$

③ $$u = u + \vec{0} \quad \text{(VS 3)}$$
$$= u + (w + (-w)) \quad \text{(VS 4)}$$
$$= (u + w) + (-w) \quad \text{(VS 2)}$$
$$= (v + w) + (-w) \quad \text{(assumption)}$$
$$= v + (w + (-w)) \quad \text{(VS 2)}$$
$$= v + \vec{0} \quad \text{(VS 4)}$$
$$= v \quad \text{(VS 3)}$$

④ $$\vec{0} + 0v = 0v + \vec{0} \quad \text{(VS 1)}$$
$$= 0v \quad \text{(VS 3)}$$
$$= (0+0)v$$
$$= 0v + 0v \quad \text{(VS 7)}$$

Cancellation law $\Rightarrow$ $0v = \vec{0}$

⑤ $$a\vec{0} + a\vec{0} = a(\vec{0} + \vec{0}) \quad \text{(VS 7)}$$
$$= a\vec{0}$$
$$= a\vec{0} + \vec{0} \quad \text{(VS 3)}$$
$$= \vec{0} + a\vec{0} \quad \text{(VS 1)}$$

Cancellation law $\Rightarrow$ $a\vec{0} = \vec{0}$

⑥ $$v + (-1)v = 1v + (-1)v \quad \text{(VS 5)}$$
$$= (1 + (-1))v \quad \text{(VS 7)}$$
$$= 0v$$
$$= \vec{0} \quad ④$$

From ②, $(-1)v = -v$

Notation

For $v, w \in V$,

$w - v$ is denoted to be $w + (-v)$